AB Calculus Facts and Formulae

Review of Trigonometry

Pythagorean Identities:

$$\sin^2\theta + \cos^2\theta = 1$$
$$\tan^2\theta + 1 = \sec^2\theta$$
$$1 + \cot^2\theta = \csc^2\theta$$

Double Angle Identities:

$$\sin 2\theta = 2\sin\theta\cos\theta$$
$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$
$$\cos 2\theta = 1 - 2\sin^2\theta$$
$$\cos 2\theta = 2\cos^2\theta - 1$$
$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$$

Limits and Continuity

What is a Limit (in words)?
"The value $f(x)$ approaches as x approaches a real # or $\pm\infty$."

One-Sided and Two-Sided Limits
If $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = L$, a real number L, then $\lim_{x\to a} f(x) = L$.

Continuity
If $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = f(a)$ (a real #), then f is continuous at $x = a$.

Intermediate Value Theorem:
If f is cont. on (a, b) and if $f(a) < k < f(b)$, then there is a c in (a, b) such that $f(c) = k$.

Extreme Value Theorem:
If f is continuous on $[a, b]$ then f attains a maximum value $f(c)$, and a minimum value $f(d)$, for some values c, d contained in $[a, b]$.

L'Hôpital's Rule:
$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$, if $\frac{0}{0}, \frac{\infty}{\infty}$. If $0 \bullet \infty$ change form to $\frac{0}{0}$ or $\frac{\infty}{\infty}$ and then use LHR.

Derivatives

Riemann Sums = werry function increasing/decreasing

Tangent line estimate = concavity

What is a Derivative (in words)?
1. An instantaneous rate of change at a point.
2. Local Linear Approximation (slope).

Four definitions of the derivative:

Top = horiz tang
Bottom = vertical tang

$$f'(x) = \lim_{h\to o}\frac{f(x+h)-f(x)}{h}$$ RDQ

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x-h)}{2h}$$

$$f'(x) = \lim_{h\to 0}\frac{f(x)-f(x-h)}{h}$$ LDQ

$$f'(c) = \lim_{x\to c}\frac{f(x)-f(c)}{x-c}$$

Differentiability
If f is continuous at $x = a$ and $\lim_{x\to a^-} f'(x) = \lim_{x\to a^+} f'(x)$, then f is differentiable at $x = a$.

Tangent Lines:

Use point-slope form: $y - f(a) = f'(a)(x - a)$

Normal Lines:

$\perp$ to the tangent of f at $x = a$, slope of normal line $= -\dfrac{1}{f'(a)}$

Derivatives of Inverses

If f and g are inverses and $f(a) = b$, then $f'(a) = \dfrac{1}{g'(b)}$. $(f^{-1})'(x) = \dfrac{1}{f'(f^{-1}(x))}$

Derivative DNE

$f'(x)$ undefined (DNE) implies graph of f has an asymptote, hole, jump or cusp.

What the derivative indicates about a monotonic function:

If $f'(x) > 0$, then f is increasing.
If $f'(x) < 0$, then f is decreasing.

Concavity

If $f''(x) > 0$, then the graph of f is concave up.
If $f''(x) < 0$, then the graph of f is concave down.

Concavity = look for slope change of $f'(t)$

look at y values to determine max point

changing sign $(f'(t))$

Points of inflection:

$f''(x)$ changes signs. Look where $f''(x) = 0$ or DNE.

Critical Values:

x-values where $f'(x) = 0$ or $f'(x)$ DNE

Absolute Extrema on a closed interval:

Evaluate function at critical values and endpoints to find maximum and minimum.

First Derivative Test for Local Extrema

a. If $f'(x)$ changes sign from negative to positive at $x = c$, then f has a local minimum at $x = c$.
b. If $f'(x)$ changes sign from positive to negative at $x = c$, then f has a local maximum at $x = c$.

Second Derivative Test for Local Extrema:

a. If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at $x = c$.
b. If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at $x = c$.

Rolle's Theorem:

If f is continuous on $[a, b]$, differentiable on (a, b), and $f(a) = f(b)$ then $f'(c) = 0$ for some c in (a, b).

Mean Value Theorem:

If f is continuous on $[a, b]$ and differentiable on (a, b), then $f'(c) = \dfrac{f(b) - f(a)}{b - a}$ for some c in (a, b).

Related Rates

Ex. $v = \pi r^2 h$

$$\frac{dv}{dt} = \pi r^2 \frac{dh}{dt} + h(2\pi r)\frac{dr}{dt}$$

Integrals

What is a Definite Integral (in words)?
1. "The net accumulation of rates of change."
2. "The limit as n approaches ∞ of a Reimann Sum"

Definite Integrals and Limits of Riemann Sums

$$\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x$$

Ex. $\int_0^{10} e^{3x}\,dx = \lim_{n\to\infty}\frac{1}{n}\left[e^{3\left(\frac{1}{n}\right)} + e^{3\left(\frac{2}{n}\right)} + \ldots + e^{3\left(\frac{10n}{n}\right)}\right]$

Approximations of Definite Integrals: Riemman Sums

n rectangles (of equal width):

$\int_a^b f(x)\,dx \approx \Delta x\left[f(x_1^*)+f(x_2^*)+\cdots+f(x_n^*)\right]$ where

$\Delta x = \dfrac{b-a}{n}$ and x_i^* is any x value in the i^{th} subinterval

(usually the left endpoint, right endpoint or midpoint)

n trapezoids (of equal height):

$\int_a^b f(x)\,dx \approx \frac{1}{2}\bullet\frac{b-a}{n}\left[f(x_0)+2f(x_1)+2f(x_2)\ldots+f(x_n)\right]$

where $x_0 = a$, $x_n = b$ and $\Delta x = \dfrac{b-a}{n}$

The Fundamental Theorem of Calculus:
If f is continuous on $[a,b]$, then

1. If $g(x)=\int_a^x f(t)\,dt$, then $g'(x)=f(x)$.
2. $\int_a^b f(x)\,dx = F(b)-F(a)$,
   where F is any antiderivative of f, that is, $F'=f$.

The Mean Value Theorem of Integrals:
(The Average Value of a Function)
If f is continuous on $[a, b]$, then there exists
c in (a, b) such that $f(c)=\dfrac{1}{b-a}\int_a^b f(x)\,dx$.

Volume by DISK/WASHER:

$V=\pi\int_{x_1}^{x_2}\left(R^2(x)-r^2(x)\right)dx$ $\qquad$ $V=\pi\int_{y_1}^{y_2}\left(R^2(y)-r^2(y)\right)dy$

R is the outer radius and r is the inner radius. Rectangles $\perp$ axis of rotation.

Volume by CROSS SECTIONS:

$\perp$ x-axis: $V = \int_{x_1}^{x_2} A(x)\,dx$ $\qquad$ $\perp$ y-axis: $V = \int_{y_1}^{y_2} A(y)\,dy$

where A is the area of the cross section.

Geometry formulas for common cross sections:

$A_{\text{equilateral}\,\triangle} = \dfrac{s^2\sqrt{3}}{4}$ $\qquad$ $A_{\text{Isosceles rgt. }\triangle\text{Leg on Base}} = \dfrac{(\text{leg})^2}{2}$ $\qquad$ $A_{\text{Isosceles Rgt. }\triangle\text{ Hyp. on Base}} = \dfrac{(\text{hypotenuse})^2}{4}$

Differential Equations

Semicircles = $\frac{\pi r^2}{2}$

Find the general solution and a particular solution.

1. Separate variables and integrate.

e.g. Solve $\dfrac{dx}{dt} = 6xt^2$ if $x(0) = 10$.

$\int \frac{dx}{x} = \int 6t^2\,dt$ — separate variables and integrate both sides

$\ln x = 2t^3 + C$ — write antiderivatives--don't forget +C!

$\ln 10 = C$ — substitute given information to solve for C

$\ln x = 2t^3 + \ln 10$ — rewrite equation

$x(t) = e^{(2t^3 + \ln 10)} = 10e^{2t^3}$ — solve for x and simplify

2. Use slope fields to approximate graph of the solution.

Motion: Horizontal or Vertical

Relationship

Position: $s(t)$ $\qquad$ Velocity: $v(t) = s'(t)$ $\qquad$ Acceleration: $a(t) = s''(t)$

Average velocity $a \le t \le b$:

$v_{avg} = \dfrac{displacement}{time} = \dfrac{\Delta s}{\Delta t} = \dfrac{s(b) - s(a)}{b - a}$ $\qquad$ (Slope of secant on graph of $s(t)$.)

Instantaneous velocity at $t = c$:

$v(c) = s'(c)$ $\qquad$ (Slope of tangent line on graph of $s(t)$.)

Speed at time $t = c$:

$|v(c)|$

Total distance

$\int_{t=a}^{t=b} |v(t)|\,dt$

Direction:

If $v(t) > 0$, then moving in the positive direction.

If $v(t) < 0$, then moving in the negative direction.

If $v(t) = 0$, the object is at rest.

Speeding Up/ Slowing Down:

If speeding up, then $v(t)$ and $a(t)$ have the same sign. Inc at inc rate / Dec at dec rate

If slowing down, then $v(t)$ and $a(t)$ have the opposite sign.

Differentiation Rules

General Formulas

$$\frac{d}{dx}(c)=0, \text{ where } c \text{ is constant}$$

$$\frac{d}{dx}(cf(x))=cf'(x)$$

$$\frac{d}{dx}[f(x)\pm g(x)]=f'(x)\pm g'(x)$$

$$\frac{d}{dx}[\overset{u}{f(x)}\bullet \overset{v}{g(x)}]=f(x)g'(x)+f'(x)g(x)$$ Product Rule

$uv' + u'v$

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]_{v}^{u}=\frac{g(x)\bullet f'(x)-f(x)\bullet g'(x)}{(g(x))^2}$$ Quotient Rule

$\dfrac{vu' - uv'}{v^2}$

$$\frac{d}{dx}[f(g(x))]=f'(g(x))g'(x)$$ Chain Rule

$$\frac{d}{dx}(x^n)=nx^{n-1}$$ Power Rule

Exponential and Logarithmic

$$\frac{d}{dx}(\log_b x)=\frac{1}{x\ln b}$$

$$\frac{d}{dx}e^x=e^x$$

$$\frac{d}{dx}b^x=b^x\ln b$$

$$\frac{d}{dx}\ln|x|=\frac{1}{x}$$

Chain Rule:

$$\frac{d}{dx}[f(g(h(x)))]=f'(g(h(x)))\bullet g'(h(x))\bullet h'(x)$$

$$\frac{dx}{dt}=\frac{dx}{dy}\frac{dy}{dz}\frac{dz}{dt}$$

Implicit Differentiation

Ex. $xy=y^2$

$$x\frac{dy}{dx}+y=2y\frac{dy}{dx}$$

$$(x-2y)\frac{dy}{dx}=-y$$

$$\frac{dy}{dx}=\frac{-y}{x-2y}$$

Trigonometric

$$\frac{d}{dx}(\sin x)=\cos x$$

$$\frac{d}{dx}(\cos x)=-\sin x$$

$$\frac{d}{dx}(\tan x)=\sec^2 x$$

$$\frac{d}{dx}(\cot x)=-\csc^2 x$$

$$\frac{d}{dx}(\sec x)=\sec x\tan x$$

$$\frac{d}{dx}(\csc x)=-\csc x\cot x$$

Inverse Trigonometric

$$\frac{d}{dx}(\sin^{-1}x)=\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\cos^{-1}x)=\frac{-1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\tan^{-1}x)=\frac{1}{1+x^2}$$

Logarithmic Differentiation

$y=\left(f(x)\right)^{g(x)}$

$\ln y = g(x)\ln\left(f(x)\right) \quad - \left\{ \text{take the log of both sides to bring down the exponent} \right.$

$\frac{1}{y}\frac{dy}{dx} = g(x)\frac{f'(x)}{f(x)} + g'(x)\ln(x) \quad - \left\{ \text{don't forget the product rule} \right.$

$$\frac{dy}{dx} = y\left[g(x)\frac{f'(x)}{f(x)} + g'(x)\ln(x)\right]$$

$$\frac{dy}{dx} = \left(f(x)\right)^{g(x)}\left[g(x)\frac{f'(x)}{f(x)} + g'(x)\ln(x)\right]$$

Integrals

Basic Properties

$\int kf(u)\,du = k\int f(u)\,du$

$\int\left[f(u)\pm g(u)\right]du = \int f(u)\,du \pm \int g(u)\,du$

u-substitution example

$\int x\sin\left(3x^2\right)dx \quad \Rightarrow \text{let } u = 3x^2$

$\Downarrow$

$\int \sin(u)\frac{1}{6}du \quad \Leftarrow du = 6x\,dx$

Basic Forms

$\int\left(u^n\right)du = \frac{u^{n+1}}{n+1} + C, n \neq -1$

$\int\frac{du}{u} = \ln|u| + C$

$\int e^u\,du = e^u + C$

$\int b^u\,du = \frac{b^u}{\ln b} + C$

Integration by Parts

$\int u\,dv = uv - \int v\,du$ LIATE

$\int \ln u\,du = u\ln u - u + C$

Trigonometric

$\int \sin u\,du = -\cos u + C$

$\int \cos u\,du = \sin u + C$

$\int \tan u\,du = -\ln|\cos u| + C$

$\int \cot u\,du = \ln|\sin u| + C$

$\int \sec u\tan u\,du = \sec u + C$

$\int \sec u\,du = \ln|\sec u + \tan u| + C$

$\int \csc u\,du = -\ln|\csc u + \cot u| + C$

$\int \sec^2 u\,du = \tan u + C$

$\int \csc^2 u\,du = -\cot u + C$

Properties of Definite Integrals

$\int_a^a f(x)\,dx = 0$

$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$